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Directed maximal partial orders of matrices ☆

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Abstract

It is proven that a directed maximal partial order on the full real matrix algebra is precisely a partial order the positive cone of which coincides with $\Pi(O)$, the set of all matrices preserving some full cone O.

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We begin with some basic concepts from the theory of cones. A survey paper that is the main reference for this topic is Barker [1].

Let us recall the definitions. A subset O of $\mathbf{R}^n$ is a *cone* if: (i) if $\alpha, \beta \geqslant 0$ and $x, y \in O$, then $\alpha x + \beta y \in O$, and (ii) $O \cap (-O) = \{0\}$. We will be concerned here with the *full* cones, i.e. such cones O that O is a topologically closed set and $\operatorname{span}(O) = \mathbf{R}^n$.

An order relation "$\leqslant$" on $\mathbf{R}^n$ is called a *partial order* on $\mathbf{R}^n$ if the relation is compatible with the linear operations (see Birkhoff [2]). We say then that $\mathbf{R}^n$ is a *partially ordered vector space*.

A cone O defines a partial order in $\mathbf{R}^n$: $x \leqslant y$ means $y - x \in O$; and vice versa, a partial order in $\mathbf{R}^n$ defines a cone, the *positive cone* of the partial order: $O = \{v \in \mathbf{R}^n : v \geqslant 0\}$.

The partial order defined by a full cone is *directed* in the sense that for any two $x, y \in \mathbf{R}^n$, there are $u, v \in \mathbf{R}^n$ such that $u \leqslant x, y \leqslant v$.

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Any element $0 \neq e \in O$ for which the conditions $v \in O$ and $e - v \in O$ imply $v = \alpha e$ for some real $\alpha \geqslant 0$ is called an *extremal* or an *extreme vector* of O. If e is an extremal of the cone O, then the set $\{\alpha e : \alpha \geqslant 0\}$ is called an *extreme ray* of the cone O. It is well known that a full cone O is the convex hull of its extreme rays. If the full cone has a finite number of extreme rays, it is called a *polyhedral* cone, and if the number equals n, the cone is called *simplicial*. The partial order defined by a simplicial cone is a lattice.

Let O be a full cone in $\mathbf{R}^n$. We will denote by $\Pi(O)$ the set of all $n \times n$ matrices such that

$$f \in \Pi(O) \Leftrightarrow f(O) \subseteq O$$

It is well known that $\Pi(O)$ is a full cone in the vector space $\mathbf{R}_n$, of all $n \times n$ matrices over $\mathbf{R}$. Hence, the partial order in $\mathbf{R}_n$ determined by $\Pi(O)$ is directed. Since also $\Pi(O) \cdot \Pi(O) \subseteq \Pi(O)$, we have that $\mathbf{R}_n$ is a *directly ordered algebra* with the positive cone $\Pi(O)$.

We will show here that if $\mathbf{R}_n$ is a partially ordered algebra with the positive cone P (the cone P satisfies $P \cdot P \subseteq P$), then the partial order is directed and maximal if and only if $P = \Pi(O)$ for some full cone O in $\mathbf{R}^n$.

The structure of $\Pi(O)$ has been studied extensively. It is well known that O is polyhedral (simplicial) if and only if $\Pi(O)$ is polyhedral (simplicial) [8,9]. Another important example of $\Pi(O)$ is based on the *ice cream cone* in $\mathbf{R}^n$ defined by $O = \left\{x \in \mathbf{R}^n : \sqrt{x_1^2 + \cdots + x_{n-1}^2} \leqslant x_n\right\}$. The structure of $\Pi(O)$ was determined by Loewy and Schneider in [5].

The following concepts were introduced and used in Ma and Wojciechowski [6]. Let $\mathbf{F}$ be a subfield of $\mathbf{R}$ and let P be a partial order of $\mathbf{F}_n$ (to mean that P is the positive cone of a partial order of the algebra $\mathbf{F}_n$). A nonempty subset S of $\mathbf{R}^n$ is said to be a P-*invariant set* if for every $f \in P$, $f(S) \subseteq S$. A cone O in $\mathbf{F}^n$ is said to be a P-*invariant cone* in $\mathbf{F}^n$ if O is also a P-invariant set. Obviously, $\{0\}$ is a P-invariant cone for every P. We will refer to this cone as to the *trivial* P-invariant cone. Other cones will be called *nontrivial*. Obviously the cone O is $\Pi(O)$-invariant (here $\Pi(O)$ is defined analogously for $\mathbf{F}_n$). The method of the P-invariant cone was crucial in the proof of the following Weinberg conjecture. Let $(\mathbf{F}_n, P)$ denote the partially ordered algebra $\mathbf{F}_n$ with the positive cone P. The conjecture has stated that if $(\mathbf{F}_n, P)$ is lattice-ordered so that the identity matrix is positive, then $(\mathbf{F}_n, P)$ is isomorphic to the usual lattice-ordered algebra of matrices (i.e. where a matrix is positive precisely when all of its entries are nonnegative numbers). The same can be expressed using the cone language: $(\mathbf{F}_n, P)$ is lattice-ordered with the identity matrix positive if and only if $P = \Pi(O)$, for some simplicial cone O in $\mathbf{F}^n$ [6]. The proof of the conjecture led, in turn, to a complete description of all lattice orders on $\mathbf{F}_n$ [7].

We recall versions of two facts proven in [6] that will be useful here:

(1) Every directed partial order P of $\mathbf{R}_n$ has a P-invariant full cone in $\mathbf{R}^n$ ([6], the statement following Lemma 3 with $\mathbf{F} = \mathbf{R}$).

(2) If P is a directed partial order on $\mathbf{R}_n$, then every closed nontrivial P-invariant cone is full ([6], Lemma 2).

We will use the following “simplicial separation lemma”.

Lemma 1. *Let O be a full cone in $\mathbf{R}^n$. Then for every $v \notin O$, there exists a simplicial cone S such that $O \subseteq S$ and $v \notin S$.*

This way a point outside the cone can be separated from the cone by a simplicial cone.

Proof. [1] By theorem 2.5 of Klee [3], there exists a linear functional f on $\mathbf{R}^n$ such that $f > 0$ on $O \setminus \{0\}$ and $f < 0$ on r, the positive ray determined by v. In fact, there is a nonempty open set of such functionals (in the conjugate space), so we can choose one whose level sets are not parallel to r. Let f be such a functional, and let the hyperplane H be defined by setting f equal to 1. Let $B = O \cap H$. Then B is a compact base for O, i.e. $O = \mathbf{R}^+ B$. Since B is a compact subset of H, B is contained in a (sufficiently large) $(n-1)$-simplex Z in H. If we let $S = \mathbf{R}^+ Z$, then S is a simplicial cone that contains O and misses v. $\square$

Proposition 2. *If O is a full cone, then O is a minimal full $\Pi(O)$-invariant cone.*[2]

Proof. Suppose that O' is a full $\Pi(O)$-invariant cone and $O' \subset O$. By Lemma 1 there is a simplicial cone S such that $O \subseteq S$. Let $v \in O \setminus O'$. Define a linear mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ by $f(e_i) = v$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ are linearly independent extremals of S. Since every vector from O is a nonnegative combination of the e_i’s, $f \in \Pi(O)$. Since O' is full, there is $w \in O' \setminus \mathrm{null}(f)$. But then $f(w) \notin O'$, a contradiction. $\square$

Proposition 3. *If O is a full cone, then O is a maximal $\Pi(O)$-invariant cone.*

Proof. Suppose that $O' \supset O$ is a $\Pi(O)$-invariant cone, and let $u \in O' \setminus O$. By Lemma 1, there is a simplicial cone S separating O from u: $O \subseteq S$ and $u \notin S$. Let $e_1, \ldots, e_n$ be independent extremals of S. Write $u = \sum_{i=1}^n \alpha_i e_i$. Since $u \notin S$, at least one of the coefficients is negative, say, $\alpha_1 < 0$. Let $0 \neq v \in O$. Define a linear mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ by

$$f(e_i) = \begin{cases} v & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

[1] This version of the proof was communicated to the author by Professor Victor Klee [4].

[2] Existence of a minimal P-invariant cone can be established for an arbitrary directed order P in $\mathbf{F}_n$. In the lattice case see [6, Theorem 8].

Since $O \subseteq S$, every vector in O is a nonnegative linear combination of the e_i's. Therefore, $f \in \Pi(O)$, and since O' is $\Pi(O)$-invariant, $f(O') \subseteq O \subset O'$. But

$$0 \neq f(u) = \alpha_1 v \in -O \subset -O'. \qquad \square$$

Contradiction.

Lemma 4. *If O_1 and O_2 are cones, then $O_1 + O_2$ is a cone if and only if $O_1 \cap -O_2 = \{0\}$.*

Proof. $O_1 + O_2$ is closed under addition and nonnegative scalar multiplication, because O_1 and O_2 are. Let now for some $u, v \in O_1 + O_2$, $u + v = 0$. Let $u = u_1 + u_2$ and $v = v_1 + v_2$ with $u_1, v_1 \in O_1$, and $u_2, v_2 \in O_2$. Then $u_1 + v_1 = -(u_2 + v_2)$. But $u_1 + v_1 \in O_1$ and $u_2 + v_2 \in O_2$, so by assumption, $u_1 + v_1 = u_2 + v_2 = 0$. Thus $u_1 = v_1 = u_2 = v_2 = 0$, so $u = v = 0$, and $O_1 + O_2$ is a cone.

The converse is obvious. $\square$

Proposition 5. *Let O and O' be full $\Pi(O)$-invariant cones. Then $O' = O$ or $O' = -O$.*

Proof. If $O + O'$ is a cone, then since $O \subseteq O + O'$, and, obviously, $O + O'$ is a full $\Pi(O)$-invariant cone, $O = O + O'$ by Proposition 3 (maximality). But then $O' \subseteq O$, so by Proposition 2 (minimality), $O' = O$.

If $O + O'$ is *not* a cone, then by Lemma 4, $O \cap -O' \neq \{0\}$, and thus $O \cap -O'$ is a nontrivial closed $\Pi(O)$-invariant cone, so by (2) it is full. Since $O \cap -O' \subseteq O$, by Proposition 2 (minimality), $O \cap -O' = O$, so $O \subseteq -O'$, and by Proposition 3 (maximality) $O = -O'$.

In other words, if O is full, then O and $-O$ are the only nontrivial $\Pi(O)$-invariant cones. $\square$

Corollary 6. *For every full cone O, $\Pi(O)$ is a maximal partial order in $\mathbf{R}_n$.*

Proof. Suppose that $\Pi(O) \subseteq P$, for some partial order P. Since P is directed, then by (1) there exists a full P-invariant cone O'. Also, $\Pi(O) \subseteq P \subseteq \Pi(O')$, so O' is $\Pi(O)$-invariant, and hence by Proposition 5, $O' = O$ or $O' = -O$. Thus we have $\Pi(O) \subseteq P \subseteq \Pi(O)$, so $P = \Pi(O)$ and $\Pi(O)$ is maximal. $\square$

Note: the closedness of O cannot be dropped as illustrated by the following example.

Example 7. Let $O = \left\{(x, y)^{\mathrm{T}} \in \mathbf{R}^2 : x > 0, y > 0\right\} \cup \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$. Then

$$\Pi(O)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}: a,b,c,d\geqslant 0;\, a^2+b^2>0,\, c^2+d^2>0\right\}\cup\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$$

But $(\mathbf{R}^+)^2$ is also $\Pi(O)$-invariant and $\Pi(O)\subset \mathbf{R}_2^+$.

Corollary 8 (Characterization of directed maximal partial orders in $\mathbf{R}_n$). *A directed partial order P is maximal if and only if $P=\Pi(O)$ for some full cone O.*

Proof. If O is full, then by Corollary 6 $\Pi(O)$ is maximal. Conversely, if P is directed, then by (1) P has a full P-invariant cone O. Since P is maximal, and $P\subseteq \Pi(O)$, we have $P=\Pi(O)$.

Finally, we notice that there exists a maximal partial order on $\mathbf{R}_n$ that is not directed, and consequently its positive cone is not $\Pi(O)$ for any full cone O. □

Example 9. Let

$$P=\left\{\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix}: y>0\right\}\cup\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$$

It is easy to see that P is a positive cone of a partial order and that this order is not directed (since P does not span $\mathbf{R}_2$). However, this ordering is maximal as it is shown in Wojciechowski and Kreinovich [10, Proposition 2.3].

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